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LETTER TO THE EDITOR

The semiclassical origin of the logarithmic singularity in the symplectic form factor

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Abstract
Sieber and Richter achieved a breakthrough towards a proof of the universality of spectral fluctuations of chaotic quantum systems conjectured by Bohigas, Giannoni and Schmidt by calculating semiclassically the first term beyond the diagonal approximation of the orthogonal form factor. In this letter, the semiclassical origin of the logarithmic singularity of the symplectic form factor is deduced perturbatively by combining this result with the contribution that arises due to the spin. This approach stands in contrast to the duality approach introduced by Bogomolny and Keating, which is essentially non-perturbative, and where the structure around the Heisenberg time is related to the structure for very small time which can be deduced using the diagonal approximation.

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1. Introduction

Numerically, it was shown for many chaotic quantum systems that the form factor is faithful to random-matrix theory [1, 2]. However, a derivation of the equivalence of the spectral properties of random-matrix theory with the properties of classically chaotic quantum systems (the so-called BGS conjecture, [3]) is still missing.

Recently, Sieber and Richter [4] have been able to make a breakthrough towards a semiclassical derivation of the BGS conjecture by calculating the first off-diagonal term of the semiclassical form factor for the orthogonal case. Using these results, we are able to proceed further by comparing the orthogonal with the symplectic symmetry class. In contrast to the bootstrap method which was introduced by Bogomolny and Keating [5] and applied to the symplectic case by Keppeler [6], we propose a generalization of the perturbative expansion introduced by Sieber and Richter.

This letter is organized as follows. First, we will derive the first off-diagonal term of the symplectic form-factor using the semiclassical spin formalism derived by Bolte and

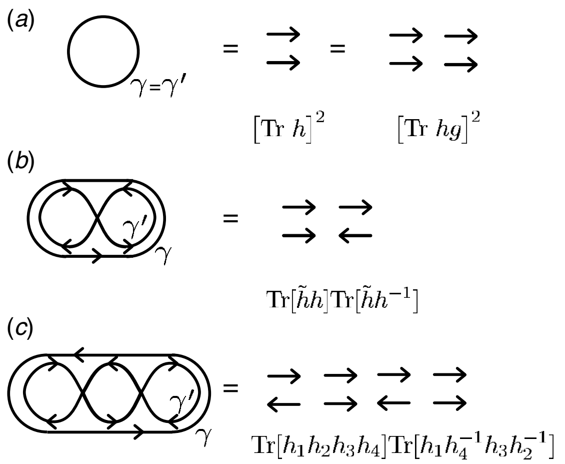

Figure 1. The spin contribution for orbits in configuration space with relative intersection number $k = 0$, 1 and 2, corresponding to equation (7), (10), (11) and (16), respectively. The matrices in the trace are $SU(2)$-matrices describing the semiclassical spin transport along the classical periodic orbits γ, γ', which are determined by the classical Hamiltonian of the corresponding spinless particle.

Keppeler [7]. Then, we proceed as follows: we also assume that the higher-order terms in the form factor can be calculated by a semiclassical expansion in terms of number of intersections, where one of the partner orbits has an intersection, the other has an avoided crossing. Already in the case of two intersection points, many different topologies are possible. However, we assume that only the topology depicted in figure 1(*c*) is relevant. Using this assumption and its generalization for higher-order contributions, we calculate the spin contribution in all orders of the semiclassical expansion for arbitrary spin j. This will lead to the final conclusion that the semiclassical expansion for the orthogonal and symplectic case are basically the same in the region below the Heisenberg time; the only differences are a scaling due to Kramers' degeneracy and a certain phase factor due to the spin. The main conclusion is that the logarithmic singularity at the Heisenberg time in the symplectic case originates from this additional phase factor.

The orthogonal form factor derived from random-matrix theory is given by [1]

$$K[\tau]_{\mathrm{orth}} = \begin{cases} 2|\tau| - |\tau|\ln[1+2|\tau|] & \text{for} \quad |\tau| < 1 \\ 2 - |\tau|\ln\dfrac{2|\tau|+1}{2|\tau|-1} & \text{for} \quad |\tau| > 1 \end{cases}$$

and has the small τ-expansion $K[\tau] = 2\tau - 2\tau^2 + 2\tau^3 \cdots$.

Let $A_\gamma \equiv T_\gamma/\sqrt{\mathrm{Tr}\,\mathrm{M}_\gamma - 2}$ be the amplitude of the classical periodic orbit γ with period T_γ and corresponding action $S(\gamma)$. In the semiclassical framework initiated by Gutzwiller [8], the form factor is then given by the following double sum over periodic orbits γ, γ':

$$K[\tau]_{\mathrm{orth}} = \lim_{\hbar\to 0} \frac{1}{2\pi\hbar\bar{d}(E)} \sum_{\gamma,\gamma'} A_\gamma A^*_{\gamma'} \exp^{\mathrm{i}(S_\gamma - S_{\gamma'})/\hbar} \delta\left(T - \frac{T_\gamma + T_{\gamma'}}{2}\right). \tag{1}$$

Here, $\bar{d}(E)$ denotes the mean density of states. Basically, the BGS conjecture states that the random-matrix theory result for the form factor can be deduced from this expression in the semiclassical limit. The semiclassical limit is defined by $\hbar \to 0$, or equivalently, $T_\gamma \to \infty$, $T_{\mathrm{H}} \to \infty$, $T_\gamma/T_{\mathrm{H}} \equiv \tau$. The leading term 2τ of the random-matrix form factor can be deduced semiclassically by taking into account the contribution of the self-correlation of all orbits. This is the so-called diagonal approximation, first derived by Berry in 1985 [9]. Sieber and Richter demonstrated that the term $-2\tau^2$ in the Taylor expansion of the random-matrix theory result can be derived semiclassically by considering a pair of orbits γ, γ' where the relative intersection number is one: the additional intersection in the first orbit defines two parts of this orbit. As compared to the orbit without additional intersection, the direction in one part of the first orbit is reversed (figure 1(*b*)). The action difference of the two orbits must be of the order of $\hbar$ in order to give a non-vanishing contribution to the form factor. Therefore, both orbits must have almost the same action; for $\hbar \to 0$, the action difference must vanish. Only at the place where the additional intersection happens, both orbits deviate from each other. These configurations do only contribute to the form factor when time-reversal symmetry is preserved, because only in this case, the time-reversed orbit exists and is given by the orginal orbit transversed in the opposite direction.

For the symplectic case, off-diagonal terms arise much the same as for the orthogonal case, because time-reversal symmetry still holds. The only difference to the orthogonal form factor is the fact that the squared time-reversal operator is negative ($T^2 = -1$) in the symplectic case. This, in turn, is realized for systems with half-integer spin. The symplectic form factor derived from random-matrix theory reads [1]

$$K[\tau]_{\mathrm{sympl}} = \begin{cases} \dfrac{|\tau|}{2} - \dfrac{|\tau|}{4} \ln |1 - |\tau|| & \text{for} \quad |\tau| < 2 \\ 1 & \text{for} \quad |\tau| > 2. \end{cases}$$

Thus, the term $+\tau^2/4$ is expected to arise from the same orbit correlation that has been considered by Sieber and Richter. The prefactor $1/4$ can be determined trivially due to Kramers' degeneracy. In the orthogonal case, energy eigenstates $|E_n\rangle,\ |TE_n\rangle$ are proportional to each other and do not lead to different quantum states. However, energy eigenstates $|E_n\rangle,\ |TE_n\rangle$ are different quantum states with the same energy due to $\langle E_n|TE_n\rangle = 0$ if $T^2 = -1$. Consider the Fourier transform of the form factor $y[e]$, which is a function of energy. We can double the energy scale $y[2e]$. In such as way, we introduce Kramers' degeneracy by hand in the orthogonal case

$$\tfrac{1}{2} K_{\mathrm{orth}}(\tfrac{1}{2}\tau) = \int_{-\infty}^{\infty} y_{\mathrm{orth}}(2e) \mathrm{e}^{2\pi \mathrm{i} e \tau}\, \mathrm{d}e. \tag{2}$$

After this scaling, the symplectic and orthogonal form factor can be compared using the same variables. The second difference between the orthogonal and the symplectic case is the contribution arising due to half-integer spin. In the symplectic case, the semiclassical first-order ($k = 1$) off-diagonal contribution corresponding to correlations between orbits where one of the partners has an avoided crossing where the other has a self-intersection with small intersection angle reads therefore

$$K^{k=1}_{\mathrm{sympl}}(\tau) = (\text{spin contribution}) * \frac{1}{2}\left(-2\left(\frac{\tau}{2}\right)^2\right). \tag{3}$$

The only non-trivial change for the symplectic case as compared to the orthogonal case is the sign factor, which is, as we intend to show, due to the spin contribution in the semiclassical form factor. It was demonstrated that the spin contribution is +1 for the diagonal part of the form factor, that is, when both orbits γ are equal [10]. However, the first off-diagonal contribution is due to the correlation between two orbits with relative intersection number one. This will lead to a sign change for the spin contribution.

2. Spin contribution for the diagonal approximation

The form factor for the symplectic case is given by [10]

$$K[\tau]_{\mathrm{sympl}} = \lim_{\hbar\to 0}\left\langle \frac{1}{2\pi\hbar\bar{d}(E)}\sum_{\gamma,\gamma'}\big(\mathrm{Tr}\ d_\gamma\ \mathrm{Tr}\ d_{\gamma'}\big)A_\gamma A^*_{\gamma'}\exp^{\mathrm{i}(S_\gamma - S_{\gamma'})/\hbar}\,\delta\left(T-\frac{T_\gamma+T_{\gamma'}}{2}\right)\right\rangle. \tag{4}$$

Comparing with the formulation for a system without spin, an additional factor Tr d_γ for each orbit arises due to the spin. In the formalism derived by Bolte and Keppeler, the classical trajectory is determined by the classical Hamiltonian of the corresponding system for the spinless particle. The degree of freedom of the spin is introduced in terms of the $SU(2)$-matrix d_γ that is transported along this classical trajectory. Let (p, x) be the starting point of a classical trajectory in phase space. Then, $\Phi^t_{\mathrm{H}}(p, x)$ describes the classical trajectory along a path γ, and M is a Hermitian and traceless matrix on the energy shell Ω_{E} in phase space describing the spin interaction. The spin is transported along the classical trajectory obeying the equation

$$\dot{d}(p, x, t) + \mathrm{i}\ M(\Phi^t_{\mathrm{H}}(p, x))d(p, x, t) = 0 \tag{5}$$

where the time derivative is understood to be along the trajectory $\Phi^t_{\mathrm{H}}(p, x)$. In [11], quantum ergodicity of the spin system has been proven under the assumption that the classical spin motion is ergodic. Basically, this relation between classical and quantum ergodicity of the spin is due to the Hopf map $\pi_{\mathrm{H}} : SU(2) \to SO(3)$ relating the classical spin motion with the quantum spin motion. In case that the classical motion of the spin is chaotic, d_γ becomes an arbitrary $SU(2)$ matrix in the semiclassical limit $t \to \infty$. In this case, $\langle \mathrm{Tr}\ d_\gamma\ \mathrm{Tr}\ d_{\gamma'}\rangle = 1$ can be concluded for the diagonal contribution, that is, for $\gamma = \gamma'$.

First, we recall the derivation of the spin contribution for the diagonal part, and then we shall show how a difference in the intersection number alters the result. The object that has to be calculated in the diagonal part of the form factor is the integral of $(\mathrm{Tr}\ d_\gamma)^2$ for large time t over the phase space consisting of the energy shell Ω_{E} and the spin group, that is, the product space $\mathcal{M} = \Omega_{\mathrm{E}} \times SU(2)$. In the semiclassical limit, we obtain

$$\langle(\mathrm{Tr}\ d_\gamma)^2\rangle = \lim_{T_\gamma\to\infty}\frac{1}{T_\gamma}\int_0^{T_\gamma}[\mathrm{Tr}\ d(\Phi^t_{\mathrm{H}}(p, x), t)]^2\,\mathrm{d}t. \tag{6}$$

Using the ergodicity theorem, the integral can be replaced by an integral over the phase space for the degree of freedom of the spin, which can be expressed as an integral over the full group $SU(2)$. Let $\mathrm{d}\mu_{\mathrm{H}}(g)$ be the Haar measure of $SU(2)$ in the variables defined by the group element g. An explicit expression for the Haar measure is given in the appendix. Changing 'coordinates' from g to gh does not affect the integration measure, that is, the Haar measure is right invariant, $\mathrm{d}\mu_{\mathrm{H}}(gh) = \mathrm{d}\mu_{\mathrm{H}}(g)$. For $T_\gamma \to \infty$, we can replace $d(\Phi^t_{\mathrm{H}}(p, x))$ by an arbitrary $SU(2)$-matrix g if the spin motion is chaotic. The result is [10]

$$\langle(\mathrm{Tr}\ d_\gamma)^2\rangle = \int_{SU(2)}[\mathrm{Tr}\ h]^2\,\mathrm{d}\mu_{\mathrm{H}}(h) = +1. \tag{7}$$

Obviously, the spin contribution is irrelevant for the diagonal part of the two-point function and the form factor.

A priori, it is not possible to calculate the spin contribution for the off-diagonal part of the form factor. However, using the result of Sieber and Richter, we know that the τ^2-term is originating from correlations between periodic orbits with relative intersection number $k = 1$. We conjecture that the τ^{k+1}-term in the Taylor expansion corresponds to orbits with relative intersection number k, and that among all possible topologies of the self-intersections on the orbit, only those depicted in figure 1(*c*) for $k = 2$, and its generalization to arbitrary k are relevant (however, at each intersection, the role of the orbits can be interchanged independently: either γ or γ' has the intersection). Then, the spin contribution can be calculated for all higher-order terms in the τ-expansion. In order to do so, we must cut one of the orbits open and reverse the direction k times, as displayed for $k = 1, 2$ in figure 1. Due to unitarity, the spin transport matrix $d({\Phi_{\mathrm{H}}}^{t+t'}(p, x), t)$ for the transport from (x, p) to ${\Phi_{\mathrm{H}}}^{t+t'}(p, x)$ is given by the matrix product of the spin transport matrix from (p, x) to ${\Phi_{\mathrm{H}}}^{t'}(p, x)$ and the spin transport matrix from ${\Phi_{\mathrm{H}}}^{t'}(p, x)$ to ${\Phi_{\mathrm{H}}}^{t+t'}(p, x)$

$$d(p, x, t + t') = d({\Phi_{\mathrm{H}}}^{t'}(p, x), t)d(p, x, t'). \tag{8}$$

This can also be verified directly by the transport equation (5). Therefore, it is possible to cut the orbit in configuration space at an arbitrary point (time t') into two pieces $d_\gamma = d_1 * d_2 = d_{\gamma'}$. First, we want to demonstrate that the spin contribution equation (7) is not modified when the classical path $x_{\mathrm{cl}}(t)$ is cut into two pieces without inverting the direction. Writing $T_\gamma = T_1 + T_2$, the spin contribution $\langle(\mathrm{Tr}\ d_\gamma)^2\rangle$ can also be expressed as $(T_2 \equiv T_\gamma - T_1)$

$$\langle(\mathrm{Tr}\ d_\gamma)^2\rangle = \lim_{T_1\to\infty}\lim_{T_2\to\infty}\frac{1}{T_1T_2}\int_0^{T_1}\mathrm{d}t'\int_{T_1}^{T_\gamma}\mathrm{d}t[\mathrm{Tr}\ d(\Phi_{\mathrm{H}}^{t'}(p, x), t)d(p, x, t')]^2. \tag{9}$$

Note that ergodicity of the spin motion in each part of the orbit can only be expected if the length of each part of the orbit becomes infinitely large. By applying twice the ergodicity theorem, a double integral over the phase space of the spin is obtained:

$$\langle(\mathrm{Tr}\ d_\gamma)^2\rangle = \int_{SU(2)}\int_{SU(2)}[\mathrm{Tr}\ hg]^2\,\mathrm{d}\mu_{\mathrm{H}}(h)\,\mathrm{d}\mu_{\mathrm{H}}(g) = \int_{SU(2)}[\mathrm{Tr}\ hg]^2\,\mathrm{d}\mu_{\mathrm{H}}(hg) = +1. \tag{10}$$

Using the right-invariance of the Haar measure, of course, the same result is obtained. That is, cutting the two orbits without reversing the direction of one of the two orbits does not alter the result, as it should be. Using the symbol S_γ for a part of the two orbits having the same direction and O_γ for a part with opposite direction, the conclusion is $S_1S_2 = S_{1+2}$ and $O_1O_2 = O_{1+2}$. In the next section, we consider the case of an orbit with one intersection, that is, an orbit with the structure S_1O_2.

3. One-intersection case

For the first off-diagonal contribution, the direction of one part of one orbit must be reversed, that is, cutting the first orbit into two parts, $d_\gamma = d_1 * d_2$, the other orbit is given by $d_{\gamma'} = d_1 * d_2^{-1}$. In the calculation of the form factor, all possible combinations of orbits have to be taken into account. Concerning the degree of freedom of the spin, consider two different orbits with the degree of freedom of the spin transported along both of them, respectively. We want to determine the number of different possibilities of relative spin position when both orbits are cut open at one point. The direction of the spin of the first orbit defines an axis. The spin direction of the spin propagating along the other orbit can have $2j+1$ different possibilities with respect to this axis. Due to ergodicity, all these directions have the same probability. In the case $j = 1/2$ and one intersection (corresponding to cutting the orbits once), 2 different states contribute to the summation over orbit pairs. We do not write explicitly the arguments of $d_1(\Phi_{\mathrm{H}}^{t'}(p, x), t)$ and $d_2(p, x, t')$ in the integral, because the only information needed for the calculation is the fact that for very large time, these matrices become arbitrary $SU(2)$-matrices, and the time integral is replaced by an integral over phase space due to the ergodicity theorem. The parametrization and explicit calculation can be found in the appendix:

$$\langle\mathrm{Tr}\ d_\gamma\ \mathrm{Tr}\ d_{\gamma'}\rangle_{1\ \mathrm{intersection}} = \lim_{T_1\to\infty}\lim_{T_2\to\infty}\frac{2}{T_1T_2}\int_0^{T_1}\mathrm{d}t'\int_{T_1}^{T_\gamma}\mathrm{d}t\ \mathrm{Tr}\ [d_1d_2]\,\mathrm{Tr}\ [d_1d_2^{-1}]$$

$$= 2\int_{SU(2)}\int_{SU(2)} \mathrm{Tr}[\tilde{h}h]\,\mathrm{Tr}[\tilde{h}h^{-1}]\,\mathrm{d}\mu_{\mathrm{H}}(h)\,\mathrm{d}\mu_{\mathrm{H}}(\tilde{h}) = -1. \tag{11}$$

Thus, we have calculated the spin contribution for the first off-diagonal term for the symplectic case. Combining with the results of Sieber and Richter and taking into account Kramers' degeneracy, the conclusion is

$$K(\tau)^{\mathrm{off}}_{\mathrm{sympl}} = \frac{1}{2}\left[-2\left(\frac{\tau}{2}\right)^2\right]\langle \mathrm{Tr}\ d_\gamma\ \mathrm{Tr}\ d_{\gamma'}\rangle_{1\ \mathrm{intersection}} = +\frac{\tau^2}{4} \tag{12}$$

as presumed.

4. Multiple-intersection case

By comparing the orthogonal and the symplectic form factor, it is possible to check whether the higher-order terms in the τ-expansion are related to correlations of orbits with relative intersection number k: assume that semiclassically, the orthogonal form factor $K(\tau)$ has been evaluated, that is, the expansion in τ has been deduced by calculating the correlation between orbits with increasing intersection number. Then, $1/2\ K(\tau/2)$ corresponds to the symplectic form factor without taking into account the spin contribution. Indeed, comparing the power expansion for $0 \leqslant \tau \leqslant 1$ of $K_{\mathrm{sympl}}(\tau)$ with $\frac{1}{2}\ K_{\mathrm{orth}}(\frac{\tau}{2})$, the result is

$$K_{\mathrm{sympl}}(\tau) = \frac{\tau}{2} + \frac{\tau^2}{4} + \frac{\tau^3}{8} + \frac{\tau^4}{12} + \frac{\tau^5}{16} + \cdots \tag{13}$$

$$\frac{1}{2}\ K_{\mathrm{orth}}\left(\frac{\tau}{2}\right) = \frac{\tau}{2} - \frac{\tau^2}{4} + \frac{\tau^3}{8} - \frac{\tau^4}{12} + \frac{\tau^5}{16} - \cdots. \tag{14}$$

The conclusion for the spin contribution for all off-diagonal terms is

$$\langle \mathrm{Tr}\ d_\gamma\ \mathrm{Tr}\ d_{\gamma'}\rangle_{\mathrm{k\ intersections}} = (-1)^k. \tag{15}$$

This equation should be interpreted as follows: taking the orbits γ, γ' corresponding to the τ^{k+1}-term of the expansion, the spin contribution must be $(-1)^k$ for half-integer spin and +1 for integer spin. *A priori*, it is not possible to calculate the spin contribution without further assumptions about the contributing orbit pairs γ, γ'. It is natural to test whether the assumption that all terms in the expansion can be calculated by considering pairs of orbits with increasing intersection number leads to the random-matrix theory result.

The spin contribution for $k = 0$ has been calculated in [10]. The contribution for spin $j = 1/2$ and orbits with relative intersection number $k = 1$ has been calculated in the previous section. Next, we consider the case where the relative intersection number is $k = 2$ and $j = 1/2$. We assume that for every new intersection, a combination same direction–opposite direction $S_\alpha O_\beta$ must be added to the existing orbit (figure 1). Summation over all possible spin contributions for the orbits leads to $(2j+1)^k = 2^k$ terms, which all give the same contribution in the semiclassical limit $T_\gamma \to \infty$.

$$\begin{aligned}\langle \mathrm{Tr}\ d_\gamma\ \mathrm{Tr}\ d_{\gamma'}\rangle_{2\ \mathrm{intersections}} &\\ = 2^2 \int_{SU(2)}\int_{SU(2)}\int_{SU(2)}\int_{SU(2)} &\mathrm{Tr}[h_1h_2h_3h_4]\,\mathrm{Tr}[h_1h_4^{-1}h_3h_2^{-1}]\\ &\mathrm{d}\mu_{\mathrm{H}}(h_1)\,\mathrm{d}\mu_{\mathrm{H}}(h_2)\,\mathrm{d}\mu_{\mathrm{H}}(h_3)\,\mathrm{d}\mu_{\mathrm{H}}(h_4) = +1.\end{aligned} \tag{16}$$

The assumption that the τ^3-term in the expansion of the form factor both in the orthogonal and symplectic case is due to correlations between orbits where the relative intersection number is two is therefore consistent with the random-matrix theory result.

Next, we give the general expression for the spin contribution in the form factor when the relative intersection number is k, and the spin $j = n/2$, n integer. The matrices h in the spin j-representation are $(2j + 1) \times (2j + 1)$ dimensional general $SU(2)$ matrices. The prefactor $(2j + 1)^k$ is due to the fact that each time when the orbit is cut open, with respect to the axis given by the first spin, the second spin can have $2j+1$ different states, which all contribute with the same weight, leading to $(2j + 1)^k$ contributing orbit pairs for in the spin j representation:

$$
\begin{aligned}
&\langle \mathrm{Tr}\ d_\gamma\ \mathrm{Tr}\ d_{\gamma'} \rangle_{\text{k intersections}} \\
&\quad = (2j+1)^k \prod_{j=1}^{2k} \int_{SU(2)_j} \mathrm{d}\mu_{\mathrm{H}}(h_j)\ \mathrm{Tr}\left[\prod_{l=1}^{2k} h_l\right] \mathrm{Tr}\left[\prod_{t=0}^{k-1} h_{2t+1} h_{2(k-t)}^{-1}\right] \\
&\quad = \begin{cases} (-1)^k & j \text{ half-integer} \\ (+1) & j \text{ integer} \end{cases}
\end{aligned} \tag{17}
$$

as presumed.

Finally, we compare $K[\tau]_{\mathrm{sympl}}$ with $1/2K[\tau/2]_{\mathrm{orth}}$ below twice the Heisenberg time and find

$$\frac{1}{2}K\left[\frac{\tau}{2}\right]_{\mathrm{orth}} = \frac{|\tau|}{2} - \frac{|\tau|}{4}\ln[1+|\tau|] \qquad |\tau| < 2 \tag{18}$$

$$K[\tau]_{\mathrm{sympl}} = \frac{|\tau|}{2} - \frac{|\tau|}{4}\ln[|1-|\tau||] \qquad |\tau| < 2. \tag{19}$$

Obviously, the sign factor which has been demonstrated to be related to the spin contribution is responsible for the singularity at the Heisenberg time $\tau = 1$ in the symplectic case. However, note that the semiclassical perturbative expansion is only valid for $0 \leqslant \tau \leqslant 1$.

To summarize, we have demonstrated that the semiclassical expansion for the off-diagonal part of the form factor in terms of correlations between the orbits γ, γ', with relative intersection number k as predicted by Sieber and Richter reproduces correctly the singularity at the Heisenberg time arising in the symplectic case. This is a strong hint for the assumption that all higher order terms of the τ-expansion of the form factor can be calculated semiclassically by considering orbit pairs with relative intersection number k. However, the calculation of the τ-expansion itself remains a challenging problem to be solved.

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Appendix

In this appendix we give the parametrization and the most important steps for the evaluation of the integrals over the group $SU(2)$. For a general $SU(2)$-matrix, the spin $j = 1/2$ representation can be expressed as ($q \equiv \exp(\mathrm{i}\eta)$, $r \equiv \exp(\mathrm{i}\xi)$)

$$h(q, r, \theta) = \begin{pmatrix} q\cos(\theta) & -r\sin(\theta) \\ r^{-1}\sin(\theta) & q^{-1}\cos(\theta) \end{pmatrix}.$$

In the spin j-representation, h can be represented as $(2j+1) \times (2j+1)$ Wigner-d-matrix. The Haar measure in terms of the variables (θ, ξ, η) reads

$$\int_{SU(2)} \mathrm{d}\mu_{\mathrm{H}}(h) = \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{4\pi^2} \sin(2\theta)\,\mathrm{d}\theta\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Now, consider an integral for the k-intersection spin $1/2$-case of the form

$$\langle \mathrm{Tr}\ d_\gamma\ \mathrm{Tr}\ d_{\gamma'}\rangle_{k\ \text{intersections}} = 2^k \prod_{j=1}^{2k} \int_{SU(2)_j} \mathrm{d}\mu_{\mathrm{H}}(h_j)\, \mathrm{Tr}\left[\prod_{t=1}^{2k} h_t\right] \mathrm{Tr}\left[\prod_{t=0}^{k-1} h_{2t+1} h^{-1}_{2(k-t)}\right]$$

$$= 2^k \prod_{j=1}^{2k} \int_{SU(2)_j} \mathrm{d}\mu_{\mathrm{H}}(h_j) f(q_j, r_j, \theta_j).$$

Expressed in terms of (q_j, r_j), the function $f(q_j, r_j, \theta_j)$ is a series $\sum_{n_j,m_j} A[n_j, m_j, \theta] \prod_{j=1}^{2k} q_j^{n_j} r_j^{m_j}$. The $4k$ integrals over (η, ξ) extract the term $A[0, 0, \theta] \equiv g(\theta)$ of $f(q_j, r_j, \theta_j)$. The remaining $2k$ integrals are then given by

$$\langle \mathrm{Tr}\ d_\gamma\ \mathrm{Tr}\ d_{\gamma'}\rangle_{k\ \text{intersections}} = 2^k \prod_{j=1}^{2k} \int_{-1}^{1} \mathrm{d}\cos(2\theta_j) \left(\frac{1}{8\pi^2}\right)^{2k} (2\pi)^{4k} g(\sin^2(\theta_j), \cos^2(\theta_j))$$

$$= 2^k \prod_{j=1}^{2k} \int_{-1}^{1} \mathrm{d}x_j \left(\frac{1}{8\pi^2}\right)^{2k} (2\pi)^{4k} g\left(\frac{(1-x_j)}{2}, \frac{(1+x_j)}{2}\right)$$

$$= (-1)^k \prod_{j=1}^{2k} \int_{-1}^{1} \mathrm{d}x_j (1-x_j) \frac{1}{(8\pi^2)^{2k}} (2\pi)^{4k} = (-1)^k.$$

The functions $f(q_j, r_j, \theta_j)$ and $g(\theta_j)$ have been determined with the help of Mathematica. For the general case of the spin j-representation, similar calculations lead to the result

$$\prod_{j=1}^{2k} \int_{SU(2)_j} \mathrm{d}\mu_{\mathrm{H}}(h_j)\, \mathrm{Tr}\left[\prod_{t=1}^{2k} h_t\right] \mathrm{Tr}\left[\prod_{t=0}^{k-1} h_{2t+1} h^{-1}_{2(k-t)}\right] = (-1)^k \frac{1}{(2j+1)^k}$$

for half-integer spin, and

$$\prod_{j=1}^{2k} \int_{SU(2)_j} \mathrm{d}\mu_{\mathrm{H}}(h_j)\, \mathrm{Tr}\left[\prod_{t=1}^{2k} h_t\right] \mathrm{Tr}\left[\prod_{t=0}^{k-1} h_{2t+1} h^{-1}_{2(k-t)}\right] = \frac{1}{(2j+1)^k}$$

for integer spin.

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